

Holomorphic Blocks for 3d Non-abelian Partition Functions

Masato Taki

*Mathematical Physics Lab., RIKEN Nishina Center,
Saitama 351-0198, Japan*

taki@riken.jp

Abstract

The most recent studies on the supersymmetric localization reveal many non-trivial features of supersymmetric field theories in diverse dimensions, and 3d gauge theory provides a typical example. It was conjectured that the index and the partition function of a 3d $\mathcal{N} = 2$ theory are constructed from a single component: the holomorphic block. We prove this conjecture for non-abelian gauge theories by computing exactly the 3d partition functions and holomorphic blocks.

1 Introduction

The pioneer work by Pestun [1] on the partition function of four-dimensional (4d) $\mathcal{N} = 2$ theories has served as a trigger to great progress on localization computation of supersymmetric gauge theories in diverse dimensions and on various manifolds [2]. Localization of three-dimensional (3d) theories is a focus of recent attention. Kapustin, Willett, and Yaakov [3, 4] extended Pestun’s idea to gauge theories on S^3 , and they obtained matrix model representations for the supersymmetric partition functions of these theories. We can solve these matrix models in large- N limit, for instance the ABJM partition function was computed by Drukker, Marino, and Putrov [5]. They found that the free energy of the ABJM theory actually shows the $N^{3/2}$ -scaling behavior which had been suggested by the AdS/CFT argument. This result is a typical example of the power of the localization approach.

The efficiency of localization reaches beyond large- N approximation. The matrix models for partition functions of $\mathcal{N} = 2$ gauge theories on S^3 was derived in [6, 7]. The integrand of this matrix model consists of a complicated combination of double-sine functions, and it looks hard on first glance to evaluate it exactly. In [8, 9], however, the authors succeeded to solve these matrix models exactly. In particular the partition functions of 3d $\mathcal{N} = 2$ $U(1)$ theories computed in [9] show the following factorization property:

$$Z^{U(1)}[S^3] = \sum_i Z_{\text{vort}}^{(i)} \tilde{Z}_{\text{anti-vort}}^{(i)}. \quad (1.1)$$

Here Z_{vort} and $\tilde{Z}_{\text{anti-vort}}$ are the K-theoretic vortex/antivortex partition function [10, 11] on $S^1 \times \mathbb{R}^2$. The summation is taken over the supersymmetric ground states which specify the vortex sector. This factorization into vortices is 3d analogue of Pestun’s expression

$$Z^{U(1)}[S^4] = \int da Z_{\text{inst}}(a) \tilde{Z}_{\text{anti-inst}}(a). \quad (1.2)$$

In this 4d case, ground states are labeled by the continuous moduli parameter a , so we take the integral over it after combining the contributions from instantons and anti-instantons. 3d factorization is therefore expected to originate from the localization after changing the way of it¹.

¹ The factorization of 2d theories was shown along the line [12, 13]

In this article we prove the factorization of this type actually occurs in non-abelian gauge theories. The matrix model for a non-abelian theory involves a complicated interaction, and so it is not easy to compute it straightforwardly. We therefore employ the Cauchy formula² and we resolve the problem into that of abelian theory. We find that the factorized partition function is consistent with the vortex/antivortex partition functions for the corresponding non-abelian theory. Our result strongly supports the conjecture [14] on the factorization of generic 3d $\mathcal{N} = 2$ theories.

This article is organized as follows. In Section 2, we review the factorization of supersymmetric partition functions and superconformal indexes of 3d $\mathcal{N} = 2$ gauge theories. In Section 3, we compute exactly the partition functions of non-abelian gauge theories based on the matrix model representation coming from localization. We then find that the partition functions are actually factorized into the holomorphic blocks. The topological string interpretation of these holomorphic blocks is given in Section 4. Section 5 is devoted to discussions of our results and future directions.

2 3d partition functions and factorization

In this section we provide a review of localization and the resulting factorization of the partition function and the superconformal index of a 3d gauge theory with $\mathcal{N} = 2$ supersymmetry. The factorization is only conjecture yet for generic $\mathcal{N} = 2$ theories, however, there exists a nice geometric interpretation of this phenomenon.

The partition functions of 3d theories were calculated with the help of supersymmetric localization. The path integral for a theory on squashed three-sphere is

$$Z = \int \mathcal{D}\Psi e^{-S[S_b^3] - t\{Q, V\}}, \quad (2.1)$$

For suitable choice of the scalar supercharge Q and the deformation action V , we can calculate it exactly in the limit $t \rightarrow \infty$ [15, 16]. As we will review in Appendix B, the partition function then becomes a kind of matrix model. The factorization of these partition functions of 3d abelian theories was found by Pasquetti in [9].

$$Z^{U(1)} = \sum_i Z_{\text{vort}}^{(i)}(q, x) \tilde{Z}_{\text{anti-vort}}^{(i)}(\tilde{q}, \tilde{x}) = \| Z_{\text{vort}} \|_{\text{S}}^2, \quad (2.2)$$

² This idea was suggested in [9].

where the operation $\widetilde{\cdot}$ in the sum acts for instance as $q = e^{\hbar} \rightarrow \tilde{q} = e^{-1/\hbar}$. So this pairing involves the S-duality transformation, and we call it the S -pairing. The geometric meaning of the S-transformation will be clear in this section.

The supersymmetric index is also important quantity to catch a part of quantum dynamics of theory. The 3d superconformal index, which is defined for a 3d SCFT, is the following trace taken over the Hilbert space of the theory on $\mathbb{R} \times S^2$:

$$I(q, z) = \text{Tr}(-1)^F e^{-\beta\{\mathcal{Q}, \mathcal{S}\}} q^{\epsilon+j} \prod_i z_i^{F_i}. \quad (2.3)$$

Here F_i is a Cartan generator of the flavor symmetry. The bosonic part of the 3d $\mathcal{N} = 2$ superconformal group is $SO(3, 2) \times SO(2)$, and the quantum numbers under the Cartan generators of its compact subgroup $SO(2)_j \times SO(3)_\epsilon \times SO(2)_R$ label the states of the 3d theory. Then the above superconformal index counts the BPS states for \mathcal{Q} and $\mathcal{Q}^\dagger = \mathcal{S}$:

$$\{\mathcal{Q}, \mathcal{S}\} = -j + \epsilon - R \equiv 0, \quad (2.4)$$

so this index does not depend on β , and we can take the limit $\beta \rightarrow \infty$ to evaluate it.

On the one hand we can write down the index as a twisted partition function of a 3d theory defined on the curved space-time $S^1 \times S^2$,

$$I(q, z) = \int_{\text{twisted b.c.}} \mathcal{D}\Psi e^{-S[S^1 \times S^2] - t\{\mathcal{Q}, V\}}. \quad (2.5)$$

The index was also calculated by using the localization formula [17, 18], and then the path integral reduces to an ordinary integral over the Cartan of the gauge group via supersymmetric localization [19]. Factorization for the resulting expression of the index was predicted in [20]

$$I^{U(N)} = \sum_i Z_{\text{vort}}^{(i)}(q, z) \cdot \overline{Z}_{\text{anti-vort}}^{(i)}(\bar{q}, \bar{z}) = \| Z_{\text{vort}} \|_{id}^2. \quad (2.6)$$

In [14, 21] this conjecture is studied in more detail.

For 3d abelian gauge theories, it is observed that this building block $Z_{\text{vort}}^{(i)}$ of the index is identical with that of the factorized partition function in [9]. The difference is the meaning of the conjugated variables \bar{q} and \bar{z} . The conjugation here is merely the inversion $\bar{q} = q^{-1}$, and we call this pairing the id -pairing.

In [14], it is proposed that the above-mentioned factorization originates in geometry on which the quantum field theory is defined. As **Figure 1**, $S^1 \times S^2$ and S^3 are id -

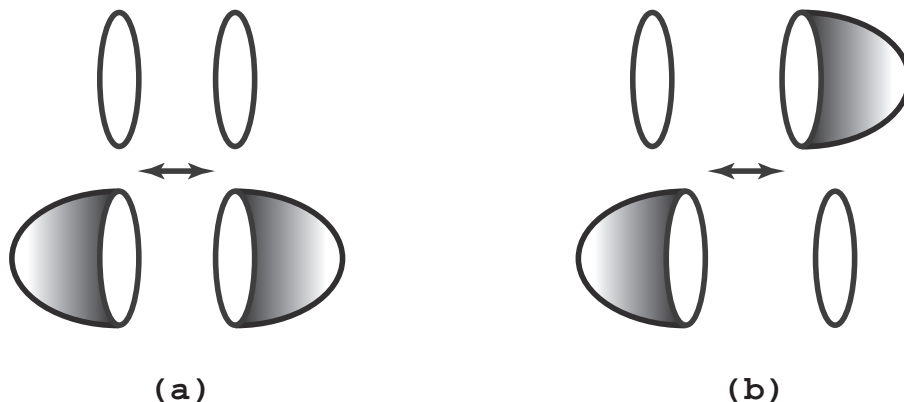


Figure 1: (a) The id-gluing of two \bar{T}^2 's: the trivial decomposition of $S^1 \times S^2$. (b) The S -gluing of two \bar{T}^2 's: the Heegaard decomposition of three-sphere S^3 through the S element of the mapping class group $SL(2, \mathbb{Z})$.

and S -gluing of a pair of solid tori $S^1 \times D^2 = \bar{T}^2$. Actually the squashed sphere S_b^3 , on which our discussion focuses, is the S -gluing of two half geometries $S^1 \times_q D^2$, and this building block is the Melvin cigar [22, 14] where D^2 fibers over S^1 with holonomy q . In [14] the authors defined the holomorphic blocks as the partition functions on this Melvin cigar $S^1 \times_q D^2$. This partition function is just the wave function for the Hilbert space on the asymptotic $\mathbb{R} \times T^2$. Here \mathbb{R} is the infinite time direction, and then a state evolves into a ground state as $\langle 0_q | i \rangle$. In this way the wave function depends on the choice of the supersymmetric vacuum $|i\rangle$ which specifies a state on the boundary T^2 . Then the gluing of two geometries through an element of $SL(2, \mathbb{Z})$ implies the following form of the partition function on the total geometry:

$$Z = \langle 0_q | 0_{\bar{q}} \rangle = \sum_i \langle 0_q | i \rangle \langle i | 0_{\bar{q}} \rangle. \quad (2.7)$$

This is the geometric origin of the factorization which was conjectured in [14].

As we mentioned, the factorization is actually observed in abelian partition functions and non-abelian superconformal indexes. So in order to verify this conjecture for a wide range of 3d theories, we have to confirm the factorization phenomenon for partition functions of non-abelian gauge theories. In the following sections, we show that non-abelian partition functions actually factorize into the holomorphic blocks which are consistent with superconformal index [21] and have a nice 3d interpretation.

3 Factorization of ellipsoid partition functions

The following matrix model gives the partition function of $\mathcal{N} = 2$ $U(N)$ gauge theory with N_f fundamentals and \bar{N}_f anti-fundamentals on squashed three-sphere [6, 7, 15, 16]

$$Z = \frac{1}{N!} \int d^N x \, e^{-i\pi k \sum x_\alpha^2 + 2\pi i \xi \sum x_\alpha} \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh \pi b(x_\alpha - x_\beta) \sinh \pi b^{-1}(x_\alpha - x_\beta) \\ \times \prod_{\alpha=1}^N \prod_{i=1}^{N_f} s_b(iQ/2 - M_i + x_\alpha) \prod_{\alpha=1}^N \prod_{j=1}^{\bar{N}_f} s_b(iQ/2 - \tilde{M}_j - x_\alpha), \quad (3.1)$$

where k is the Chern-Simons coupling, ξ is the FI-parameter for the $U(1)$ factor, and b is the squashing parameter for the three-sphere. We give masses for fundamental and anti-fundamental chiral field. When $N_f = \bar{N}_f$, this theory is the mass deformation of $\mathcal{N} = 3$ SQCD and a pair of a fundamental and an anti-fundamental chirals forms a hypermultiplet of the $\mathcal{N} = 3$ theory. The deformation of $M = \tilde{M}$ type is the vector mass for the original hypermultiplet, and $M = -\tilde{M}$ type is the axial mass. In the following, we turn on these mass deformations.

3.1 $\mathcal{N} = 2$ $U(N)$ vector-like theory

We start with studying $U(N)$ non-chiral gauge theory whose matter content consists of $\mathcal{N} = 2$ mass deformation of $\mathcal{N} = 3$ hypermultiplets. The localized partition function is

$$Z = \frac{1}{N!} \int d^N x \, e^{-i\pi k \sum x_\alpha^2 + 2\pi i \xi \sum x_\alpha} \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh \pi b(x_\alpha - x_\beta) \sinh \pi b^{-1}(x_\alpha - x_\beta) \\ \times \prod_{\alpha=1}^N \prod_{i=1}^{N_f} \frac{s_b(x_\alpha + m_i + \mu_i/2 + iQ/2)}{s_b(x_\alpha + m_i - \mu_i/2 - iQ/2)}. \quad (3.2)$$

For $k = 0$ we can enclose the integral contour in the upper half-plane as **Figure 2**, and in [9] the author employed it for generic Chern-Simons coupling. In this paper we follow the argument there.

To evaluate the matrix model for non-abelian theory $N \geq 2$, the best strategy is to translate this “many-body” problem into that of abelian theory, or a collection of non-interacting one-body systems. This idea of “abelianization” plays a role in solving many problems in for instance [8, 23], and the Cauchy formula is the key in these articles. The

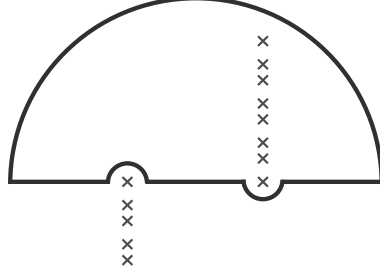


Figure 2: The integral contour for the partition function. The crosses denote the poles of the integrand of the partition function.

Cauchy formula implies

$$\begin{aligned} & \prod_{1 \leq \alpha < \beta \leq N} 2 \sinh(x_\alpha - x_\beta) \\ &= \frac{1}{\prod_{1 \leq \alpha < \beta \leq N} 2 \sinh(\chi_\alpha - \chi_\beta)} \sum_{\sigma \in S^N} (-1)^\sigma \prod_{\alpha} \prod_{\beta \neq \sigma(\alpha)} 2 \cosh(x_\alpha - \chi_\beta), \end{aligned} \quad (3.3)$$

for auxiliary variables $\chi_\alpha \not\equiv \chi_\beta \pmod{\pi i}$. So with this formula, we can resolve the “sinh-interaction” between “particles” x_α into a collection of one-particle systems in a background χ_α . We therefore use this formula as a separation of variables of our problem.

Substituting the formula (3.3) into the partition function, we obtain the following expression:

$$Z = \frac{1}{N! \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh \pi b(\chi_\alpha - \chi_\beta) \sinh \pi b^{-1}(\chi'_\alpha - \chi'_\beta)} \sum_{\sigma, \rho \in S^N} (-1)^{\sigma+\rho} \prod_{\alpha} Z[\sigma(\alpha), \rho(\alpha)], \quad (3.4)$$

$$\begin{aligned} Z[\sigma(\alpha), \rho(\alpha)] &= \oint dx \, e^{-i\pi k x^2 + 2\pi i \xi x} \prod_{\beta \neq \sigma(\alpha)} 2 \cosh \pi b(x - \chi_\beta) \prod_{\beta \neq \rho(\alpha)} 2 \cosh \pi b^{-1}(x - \chi'_\beta) \\ &\quad \times \prod_{i=1}^{N_f} \frac{s_b(x_\alpha + m_i + \mu_i/2 + iQ/2)}{s_b(x_\alpha + m_i - \mu_i/2 - iQ/2)}. \end{aligned} \quad (3.5)$$

Therefore the evaluation of the abelian integral $Z[\sigma(\alpha), \rho(\alpha)]$ immediately implies an explicit formula for the non-abelian partition function. This integral is essentially equal to that was computed in [9], and we provide an explicit computation in Appendix C. By computing this integral exactly, we find the following factorized form of the partition function of the 3d non-chiral theory:

$$Z = \frac{1}{N!} \sum_{i_1=1}^{N_f} \cdots \sum_{i_N=1}^{N_f} Z_{\text{cl}}^{\{i_\alpha\}} Z_{\text{pert}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \tilde{Z}_V^{\{i_\alpha\}}. \quad (3.6)$$

Here the summation is taken over the sequence of integers $\{i_\alpha = 1, \dots, N_f\}$, which labels the supersymmetric ground states of the theory on $S^1 \times \mathbb{R}^2$. The perturbative part is given by

$$Z_{\text{cl}}^{\{i_\alpha\}}(m, \mu, \xi) = \prod_{\alpha=1}^N e^{-i\pi k(m_{i_\alpha} + \mu_{i_\alpha}/2)^2 - 2\pi i \xi(m_{i_\alpha} + \mu_{i_\alpha}/2)}, \quad (3.7)$$

$$Z_{\text{pert}}^{\{i_\alpha\}}(m, \mu, b) = \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh(\pi b D_{i_\alpha i_\beta}) 4 \sinh(\pi b^{-1} D_{i_\alpha i_\beta}) \prod_{\alpha=1}^N \frac{\prod_{j \neq i_\alpha}^{N_f} s_b(D_{ji_\alpha} + iQ/2)}{\prod_{j=1}^{N_f} s_b(C_{ji_\alpha} - iQ/2)}, \quad (3.8)$$

where

$$D_{ji} = m_j - m_i + \mu_j/2 - \mu_i/2, \quad C_{ji} = m_j - m_i - \mu_j/2 - \mu_i/2. \quad (3.9)$$

The remaining parts, which are the holomorphic blocks, take the form

$$\begin{aligned} Z_V^{\{i_\alpha\}} &= \sum_{m_1, \dots, m_N=0}^{\infty} \prod_{\alpha=1}^N \left((-1)^N e^{\pi b \sum \mu_j} q^{N_f/2} z_\alpha \right)^{m_\alpha} q^{-k m_\alpha^2/2} \\ &\times \prod_{\alpha=1}^N \prod_{l=1}^{m_\alpha} \frac{\prod_{j=1}^{N_f} 2 \sinh \pi b (C_{ji_\alpha} + i(l-1)b)}{\prod_{\beta=1}^N 2 \sinh \pi b (D_{i_\alpha i_\beta} + i(l-1-m_\alpha)b) \prod_{j=1, j \neq i_\alpha}^{N_f} 2 \sinh \pi b (D_{ji_\alpha} + i l b)}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \tilde{Z}_V^{\{i_\alpha\}} &= \sum_{n_1, \dots, n_N=0}^{\infty} \prod_{\alpha=1}^N \left((-1)^N e^{\pi b^{-1} \sum \mu_j} \tilde{q}^{N_f/2} \tilde{z}_\alpha \right)^{n_\alpha} \tilde{q}^{-k n_\alpha^2/2} \\ &\times \prod_{\alpha=1}^N \prod_{l=1}^{n_\alpha} \frac{\prod_{j=1}^{N_f} 2 \sinh \pi b^{-1} (C_{ji_\alpha} + i(l-1)b^{-1})}{\prod_{\beta=1}^N 2 \sinh \pi b^{-1} (D_{i_\alpha i_\beta} + i(l-1-n_\alpha)b^{-1}) \prod_{j=1, j \neq i_\alpha}^{N_f} 2 \sinh \pi b^{-1} (D_{ji_\alpha} + i l b^{-1})}. \end{aligned} \quad (3.11)$$

These blocks are precisely equal to the 3d (K-theoretic) uplift of the vortex and anti-vortex partition functions [10] for $U(N)$ gauge theory with N_f antifundamental and $N_f - N$ fundamental chiral multiplets. The Coulomb branch and the mass parameters for the vortex theory are

$$a_\alpha = m_{i_\alpha} + \frac{\mu_{i_\alpha}}{2}, \quad \alpha = 1, \dots, N \quad (3.12)$$

$$M_j = m_j - \frac{\mu_j}{2}, \quad j = 1, \dots, N_f \quad (3.13)$$

$$\bar{M}_j = m_j + \frac{\mu_j}{2} + i b^\pm, \quad j \notin \{i_1, \dots, i_N\}. \quad (3.14)$$

See Appendix C for detailed computation and discussion.

The classical and 1-loop part $Z_{\text{cl}}^{\{i_\alpha\}} Z_{\text{pert}}^{\{i_\alpha\}}$ is basically a product of those of $U(1)$ theory, and we can show that it is precisely the perturbative part of the vortex/antivortex partition function [10]. Actually, we have the factorization of the 1-loop contributions [9]

$$Z_{\text{pert}}^{\{i_\alpha\}} = \prod_{\alpha} e^{i\pi \sum_j ((D_{ji_\alpha} + iQ/2)^2 - (C_{ji_\alpha} - iQ/2)^2)/2} \times Z_{\text{1-loop}}^{\{i_\alpha\}} \tilde{Z}_{\text{1-loop}}^{\{i_\alpha\}}, \quad (3.15)$$

where

$$Z_{\text{1-loop}}^{\{i_\alpha\}} = \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh(\pi b D_{i_\alpha i_\beta}) \prod_{\alpha} \prod_{\ell=1}^{\infty} \frac{\prod_{j \neq i_\alpha} (1 - q^\ell e^{-2\pi b D_{ji_\alpha}})}{\prod_j (1 - q^{\ell-1} e^{-2\pi b C_{ji_\alpha}})}, \quad (3.16)$$

$$\tilde{Z}_{\text{1-loop}}^{\{i_\alpha\}} = \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh(\pi b^{-1} D_{i_\alpha i_\beta}) \prod_{\alpha} \prod_{\ell=1}^{\infty} \frac{\prod_{j \neq i_\alpha} (1 - \tilde{q}^\ell e^{-2\pi b^{-1} D_{ji_\alpha}})}{\prod_j (1 - \tilde{q}^{\ell-1} e^{-2\pi b^{-1} C_{ji_\alpha}})}. \quad (3.17)$$

The prefactor $\prod_{\alpha} e^{\pi i \sum (D+iQ/2)^2 - (C-iQ/2)^2}$ can be absorbed into the classical part, up to irrelevant overall coefficient $e^{-i\pi N \sum m_j \mu_j}$, by changing the FI parameters [9]

$$\xi \rightarrow \xi_{\text{eff}} = \xi + \frac{1}{2} \sum_j (\mu_j + iQ). \quad (3.18)$$

We thus obtain the factorization of the non-chiral $U(N)$ theory as a natural extension of that of $U(1)$ theory:

$$Z = \frac{1}{N!} \sum_{i_1=1}^{N_f} \cdots \sum_{i_N=1}^{N_f} Z_{\text{cl}}^{\{i_\alpha\}}(\xi_{\text{eff}}) \left(Z_{\text{1-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \right) \left(\tilde{Z}_{\text{1-loop}}^{\{i_\alpha\}} \tilde{Z}_V^{\{i_\alpha\}} \right). \quad (3.19)$$

As we had expected, the holomorphic block $Z_{\text{1-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}}$ coming from this factorization coincides with that of the superconformal index of the same gauge theory [21]. We can therefore conclude that the single holomorphic block leads to not only the partition function but also the superconformal index of the vector-like gauge theory. As we will see in below, this fact holds for chiral theories.

In the next section, we will see that this non-abelian holomorphic block $Z_{\text{1-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}}$ we derived here can be reformulated into an open topological string partition function in the presence of N A-branes on strip geometry.

3.2 $\mathcal{N} = 2$ $U(N)$ chiral theory

We move on to studying chiral gauge theory. In this section we deal with $\mathcal{N} = 2$ $U(N)$ gauge theory with $2N_f$ fundamental chiral multiplets. The partition function is given

by the following matrix model:

$$Z = \frac{1}{N!} \int d^N x \, e^{-i\pi k \sum x_\alpha^2 + 2\pi i \xi \sum x_\alpha} \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh \pi b (x_\alpha - x_\beta) \sinh \pi b^{-1} (x_\alpha - x_\beta) \\ \times \prod_{\alpha=1}^N \prod_{i=1}^{2N_f} s_b(x_\alpha + \mu_i/2 + iQ/2). \quad (3.20)$$

Here we turn on the axial masses μ_i to the chiral multiplets by turning on the scalar VEVs for the background vector multiplets of weakly-gauged $U(1)$ symmetry.

This partition function also takes the factorized form

$$Z = \frac{1}{N!} \sum_{i_1=1}^{2N_f} \cdots \sum_{i_N=1}^{2N_f} Z_{\text{cl}}^{\{i_\alpha\}} Z_{\text{pert}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \tilde{Z}_V^{\{i_\alpha\}}. \quad (3.21)$$

See Appendix C for detailed computation. The perturbative part is given by

$$Z_{\text{cl}}^{\{i_\alpha\}}(m, \mu, \xi) = \prod_{\alpha=1}^N e^{-i\pi k (\mu_{i_\alpha}/2)^2 - 2\pi i \xi \mu_{i_\alpha}/2}, \quad (3.22)$$

$$Z_{\text{pert}}^{\{i_\alpha\}}(m, \mu, b) = \prod_{\alpha < \beta} 4 \sinh \pi b (E_{i_\alpha i_\beta}) \sinh \pi b^{-1} (E_{i_\alpha i_\beta}) \\ \times \prod_{\alpha=1}^N \prod_{j \neq i_\alpha}^{2N_f} s_b(E_{ji_\alpha} + iQ/2) \propto Z_{1\text{-loop}}^{\{i_\alpha\}} \tilde{Z}_{1\text{-loop}}^{\{i_\alpha\}}, \quad (3.23)$$

where $E_{ji} = \mu_j/2 - \mu_i/2$. The proportional coefficient in the last line can be absorbed into the classical part by the change of the couplings

$$k \rightarrow k_{\text{eff}} = k + N_f, \quad \xi \rightarrow \xi_{\text{eff}} = \xi + \frac{iQ N_f}{2}. \quad (3.24)$$

The full partition function then take the following factorized form

$$Z = \frac{1}{N!} \sum_{i_1=1}^{N_f} \cdots \sum_{i_N=1}^{N_f} Z_{\text{cl}}^{\{i_\alpha\}}(k_{\text{eff}}, \xi_{\text{eff}}) \left(Z_{1\text{-loop}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \right) \left(\tilde{Z}_{1\text{-loop}}^{\{i_\alpha\}} \tilde{Z}_V^{\{i_\alpha\}} \right). \quad (3.25)$$

The holomorphic block for this chiral theory is

$$Z_V^{\{i_\alpha\}} = \sum_{m_1, \dots, m_N=0}^{\infty} \prod_{\alpha=1}^N ((-1)^N q^{N_f} z_\alpha)^{m_\alpha} q^{-k m_\alpha^2/2} \\ \times \prod_{\alpha=1}^N \prod_{l=1}^{m_\alpha} \frac{1}{\prod_{\beta=1}^N 2 \sinh \pi b (E_{i_\alpha i_\beta} - i l b) \prod_{j=1, j \neq i_\alpha}^{2N_f} 2 \sinh \pi b (E_{ji_\alpha} + i l b)}. \quad (3.26)$$

The anti-vortex part $\tilde{Z}_V^{\{i\}}$ is given by the replacement $b \rightarrow b^{-1}$, $q \rightarrow \tilde{q}$ and $z_\alpha \rightarrow \tilde{z}_\alpha$. This is precisely equal to the vortex partition function for $U(N)$ theory with $2N_f - N$ fundamental chiral multiplets. This is the non-abelian generalization of the holomorphic block for $N = 1$ theory [9].

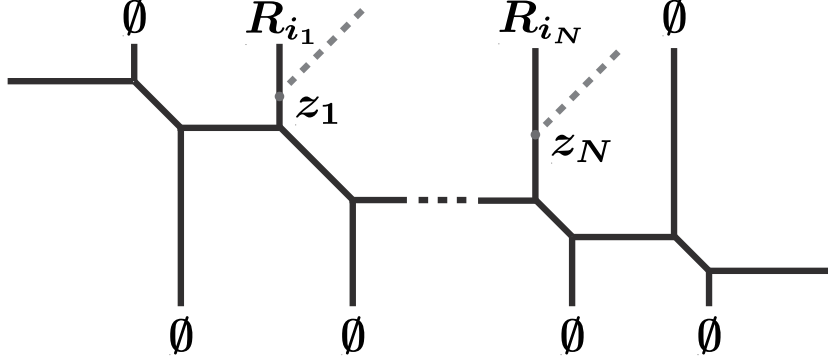


Figure 3: This strip geometry is half of the toric geometry which leads to 4d $\mathcal{N} = 2$ $U(N_f)$ gauge theory with $2N_f$ flavors through the geometric engineering. We set $R_j = \emptyset$ for $j \notin \{i_\alpha\}$.

4 Vortex partition functions and topological strings

In this section, we provide an interpretation of non-abelian vortex partition functions in terms of open topological string theory. In [1], it was shown that the holomorphic blocks for $U(1)$ gauge theories are given by partition functions of open topological strings in the presence of single A-brane. By generalizing this argument, we demonstrate that the open topological string partition functions with multiple A-branes give the holomorphic blocks for non-abelian gauge theories.

The topological string partition function [24] of the strip geometry **Figure 4** which gives the Nekrasov partition function of 4d $\mathcal{N} = 2$ gauge theory with $N_F = 2N_C$ flavors is [25]

$$\frac{\mathcal{K}_{\emptyset\emptyset\cdots\emptyset}^{R_1R_2\cdots R_N}}{\mathcal{K}_{\emptyset\emptyset\cdots\emptyset}} = \prod_{i=1}^N S_{R_i}(q^\rho) \prod_{l=1}^{\infty} \frac{\prod_{i \leq j} (1 - q^l Q_{a_i b_j})^{C_l(R_i, \emptyset)} \prod_{j < i} (1 - q^l Q_{b_j a_i})^{C_l(\emptyset, R_i^T)}}{\prod_{i < i'} (1 - q^l Q_{a_i a_{i'}})^{C_l(R_i, R_{i'}^T)}}, \quad (4.1)$$

where

$$\sum_l C_l(Y, R) q^l = q^{-1} (q-1)^2 f_Y(q) f_R(q) + f_Y(q) + f_R(q), \quad f_Y(q) = \sum_{(i,j) \in Y} q^{j-i}. \quad (4.2)$$

In the following we show that the open string partition function on this geometry gives the holomorphic block for $U(N)$ non-chiral theory in the previous section. Notice that this geometry is the same as that of $U(1)$ case [1]. The difference is the number of A-branes we insert in the geometry, and we now consider N branes for non-abelian gauge theory. Let us consider A-brane insertion at i_α -th legs of the strip geometry as **Figure 4**.

Since the world-sheet instanton on single A-brane is labeled by the Young diagrams 1^m , the following assignment of representations leads to the open string partition function with instanton mode m_α :

$$R_{i_\alpha} = 1^{m_\alpha} \text{ for } i_\alpha=1,2,\dots,N, \quad \text{otherwise } R_j = \emptyset. \quad (4.3)$$

Let t_α be the open string moduli on the i_α -th brane. The open topological string partition function $Z = \sum_{\{i_\alpha\}} e^{\sum t_\alpha m_\alpha} Z^{\{m_\alpha\}}$ is given by the following partition function of strip geometry with non-trivial representations $[1^{m_\alpha}]$:

$$\begin{aligned} Z^{\{m_\alpha\}} &\equiv \frac{\mathcal{K}_{\emptyset\emptyset\ldots\emptyset}^{\emptyset\ldots 1^{m_1}\ldots 1^{m_N}\ldots\emptyset}}{\mathcal{K}_{\emptyset\emptyset\ldots\emptyset}^{\emptyset\emptyset\ldots\emptyset}} \\ &= \prod_{\alpha=1}^N \prod_{l=1}^{m_\alpha} \frac{1}{1-q^l} \prod_{l=1}^{m_\alpha} \frac{\prod_{i_\alpha \leq j} (1-q^{l-1} Q_{a_{i_\alpha} b_j}) \prod_{j < i_\alpha} (1-q^{l-1} Q_{b_j a_{i_\alpha}})}{\prod_{i_\alpha < i_\beta} (1-q^{l-1} Q_{a_{i_\alpha} a_{i_\beta}}) \prod_{i_\alpha > i_\beta} (1-q^{l-1} Q_{a_{i_\beta} a_{i_\alpha}})} \\ &\quad \times \frac{1}{\prod_{j > i_\alpha, \notin \{i_\alpha\}} (1-q^{l-1} Q_{a_{i_\alpha} b_j}) \prod_{j < i_\alpha, \notin \{i_\alpha\}} (1-q^{l-1} Q_{b_j a_{i_\alpha}})} \\ &= \prod_{\alpha=1}^N \prod_{l=1}^{m_\alpha} \frac{1}{1-q^l} \prod_{l=1}^{m_\alpha} \frac{\prod_j (1-q^{l-1} Q_{a_{i_\alpha} b_j})}{\prod_{i_\alpha, i_\beta} (1-q^{l-1} Q_{a_{i_\alpha} a_{i_\beta}}) \prod_{j \notin \{i_\alpha\}} (1-q^{l-1} Q_{a_{i_\alpha} b_j})}. \end{aligned} \quad (4.4)$$

Under the identification between parameters,

$$Q_{a_{i_\alpha} b_j} = e^{-2\pi b C_{j i_\alpha}}, \quad Q_{b_j a_{i_\alpha}} = q e^{-2\pi b D_{j i_\alpha}}, \quad Q_{a_{i_\alpha} a_{i_\beta}} = e^{-2\pi b D_{i_\alpha i_\beta}}, \quad (4.5)$$

this open string partition function is precisely the holomorphic block for 3d $U(N)$ vector-like theory up to an overall monomial which is not relevant for our discussion. In other words, the open topological string partition function on the strip geometry gives the K-theoretic uplift of the vortex partition function for $U(N)$ theory with N_f antifundamentals and $N_f - N$ fundamentals. This is a generalization of the relation between the vortex partition function and topological strings found in [26] where the authors studied the special case $N = N_f$.

It is straightforward to generalize this computation for $U(N)$ chiral theory. The relevant geometry for this case is the half $SU(2N_f)$ geometry, and this geometry is the same as that of $U(1)$ case [1] again. We skip the detailed computation since it is merely a slight modification of the above case, but it is easy to see that the corresponding partition function gives the holomorphic block for the $U(N)$ gauge theory with $2N_f$ fundamentals.

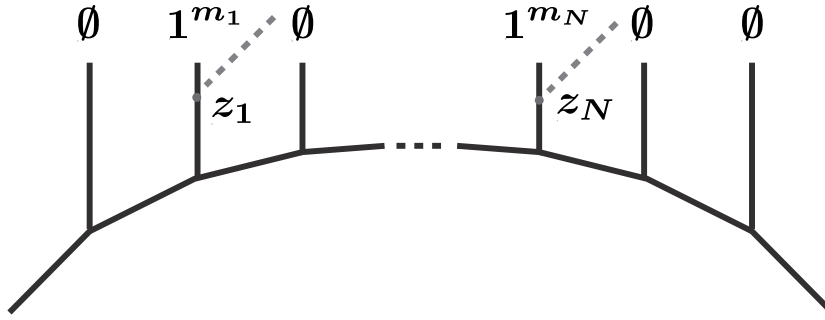


Figure 4: This strip geometry is half of the toric geometry which leads to 4d $\mathcal{N} = 2$ $U(2N_f)$ pure Yang-Mills theory through the geometric engineering.

5 Discussion

In this article, we computed exactly the supersymmetric partition functions of $\mathcal{N} = 2$ $U(N)$ gauge theories on the squashed three-sphere. We then found that the resulting expression shows the factorized structure, and it leads to the expected holomorphic block of the 3d theory. In this way we gave an explicit proof of the factorization conjecture for a range of non-abelian gauge theories. The obtained holomorphic blocks are consistent with the computation of the superconformal index, and we found that the blocks can be recast into open topological string partition functions with N A-branes.

The system of the vortex counting of a 2d theory coupled to a bulk 4d $\mathcal{N} = 2$ gauge theory describes the surface operator of the 4d theory [26, 27, 11, 28, 29, 30, 31, 32]. Since the 3d uplift of the vortex theory gives the holomorphic block, it is very interesting to investigate the 5d uplift of the surface operator, which is 3d gauge theory coupled to 5d gauge theory. In this way we can synthesize the factorizations of 4d theory (1.2) and 3d theory (1.1), and then this interplay of them should lead to new phenomenon in 5d. From the perspective of the AGT correspondence, q -Toda theory will play a role in this direction [33, 28, 34]. It should be interesting to study this issue further.

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Appendix A Double-sine function

In [6], Jafferis found that the following ℓ -function plays a role in the localization computation and the F-extremization of three-dimensional theories:

$$\ell(z) = -z \log(1 - e^{2\pi iz}) + \frac{i}{2} \left(\pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi iz}) \right) - \frac{i\pi}{12}, \quad (\text{A.1})$$

whose defining property is

$$\frac{d\ell}{dz} = -\pi z \cot \pi z, \quad \ell(0) = 0. \quad (\text{A.2})$$

Let us consider the function $s(x) = e^{\ell(-ix)}$. This function satisfies many nice properties, and we can show that this is a specialization of the double-sine function s_b :

$$s(x) = s_{b=1}(x). \quad (\text{A.3})$$

The double-sine function is defined as a natural extension of the sine function through the product expression [35]

$$s_b(x) = \prod_{m,n=0}^{\infty} \frac{mb + n/b + Q/2 - ix}{mb + n/b + Q/2 + ix}. \quad (\text{A.4})$$

We can recast this definition into the language of Barnes gamma function $\Gamma_b(x) = \Gamma_2(x|b, b^{-1})$ [36]

$$s_b(x) = \frac{\Gamma_b(Q/2 + ix)}{\Gamma_b(Q/2 - ix)}. \quad (\text{A.5})$$

From the definition, we can easily see the inversion relation $s_b(x) = 1/s_b(-x)$.

The double-sine function is meromorphic, and it satisfies the following properties

$$\begin{aligned} \frac{s_b(x + iQ/2 + imb + in/b)}{s_b(x + iQ/2)} &= \frac{(-1)^{mn}}{\prod_{k=1}^m 2i \sinh \pi b(x + ikb) \prod_{\ell=1}^n 2i \sinh \pi b(x + i\ell/b)} \\ &= \frac{(-1)^{mn} (-i)^{m+n} q^{m(m+1)/4} \bar{q}^{n(n+1)/4} e^{-\pi b m x} e^{-\pi n x/b}}{\prod_{k=1}^m (1 - q^k e^{-2\pi b x}) \prod_{\ell=1}^n (1 - \tilde{q}^\ell e^{-2\pi x/b})}, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} \frac{s_b(x - iQ/2 + imb + in/b)}{s_b(x - iQ/2)} &= \frac{(-1)^{mn}}{\prod_{k=1}^m 2i \sinh \pi b(x - iQ + ikb) \prod_{\ell=1}^n 2i \sinh \pi b(x - iQ + i\ell/b)} \\ &= \frac{(-1)^{mn} (-i)^{m+n} q^{m(m+1)/4} \bar{q}^{n(n+1)/4} e^{-\pi b m (x - iQ)} e^{-\pi n (x - iQ)/b}}{\prod_{k=1}^m (1 - q^{k-1} e^{-2\pi b x}) \prod_{\ell=1}^n (1 - \tilde{q}^{\ell-1} e^{-2\pi x/b})}, \end{aligned} \quad (\text{A.7})$$

where we introduce

$$q = q_1^{-2} = e^{-2\pi i b^2}, \quad \tilde{q} = q_2^{-2} = e^{-2\pi i/b^2}. \quad (\text{A.8})$$

These formulas come from the following expression of this function:

$$s_b(x) = \frac{e^{-i\pi/2x^2}}{\prod_{k=1}^{\infty} (1 - q^{k-1/2} e^{-2\pi b x}) \prod_{\ell=1}^{\infty} (1 - \tilde{q}^{\ell-1/2} e^{-2\pi x/b})}. \quad (\text{A.9})$$

We also have the integral representation

$$\log s_b(ix) = \int_0^{\infty} \frac{dt}{t} \left(\frac{\sinh 2tx}{2 \sinh bt \sinh t/b} - \frac{x}{t} \right). \quad (\text{A.10})$$

The residue is $\text{Res}_{x=iQ/2} s_b(x) = 1/2\pi$.

Appendix B 3d partition function

The supersymmetric localization of gauge theories on S^3 enables us to compute their partition functions as the conventional matrix model over the gauge group [6, 7]:

$$Z = \frac{1}{|\mathcal{W}|} \int \prod_H d\sigma Z^{\text{cl}}(\sigma) Z_{1\text{-loop}}^{\text{vector}}(\sigma) Z_{1\text{-loop}}^{\text{chiral}}(\sigma). \quad (\text{B.1})$$

Here the “dynamical” variable $\sigma = \sum \sigma_i H_i$ originates from the auxiliary scalar component of the vector multiplet. The localization reduces the path integral onto the constant VEV of the scalars. Then the saddle point approximation is exact, and the one-loop computation provides the factor $Z_{1\text{-loop}}(\sigma)$

$$Z_{1\text{-loop}}^{\text{vector}}(\sigma) = \det_{\text{Ad}} \frac{2 \sinh \pi \sigma}{\pi \sigma} = \prod_{\alpha \in \Delta_+} \frac{(2 \sinh \pi \alpha_i \sigma_i)^2}{(\pi \alpha_i \sigma_i)^2}. \quad (\text{B.2})$$

The denominator cancels with the Vandermonde determinant when we replace the matrix integral with the eigenvalue integral $\int \prod_H d\sigma \rightarrow \int \prod_i d\sigma_i$.

The chiral multiplet in the representation R with R-charge (conformal dimension) q gives the contribution

$$Z_{1\text{-loop}}^{\text{chiral}}(\sigma) = \det_R e^{\ell(1-q+i\sigma)} = \prod_{\rho \in R} s_{b=1}(i - iq - \rho_i \sigma_i). \quad (\text{B.3})$$

These results are generalized to the theories on the squashed sphere S_b^3 [15, 16]. There are some realizations of squashed three-sphere S_b^3 , and each preserves different

subgroup of the isometry of S^3 . In this article we adopt that of the last section of [15], which is natural in our context. The building blocks of the partition functions then receive a slight modification by the squashing parameter b :

$$Z_{1\text{-loop}}^{\text{vector}}(\sigma) = \prod_{\alpha \in \Delta_+} \frac{4 \sinh \pi b \alpha_i \sigma_i \sinh \pi b^{-1} \alpha_i \sigma_i}{(\pi \alpha_i \sigma_i)^2}, \quad (\text{B.4})$$

$$Z_{1\text{-loop}}^{\text{chiral}}(\sigma) = \prod_{\rho \in R} s_b(iQ/2 - iq - \rho_i \sigma_i). \quad (\text{B.5})$$

Appendix C Details of computation

Appendix C.1 vector-like theory

In this section we provide detailed computation of the partition function (3.4) of $\mathcal{N} = 2$ non-chiral gauge theory. For the purpose, we start with computing (3.5). Recall that the simple poles and zeros of the double-sine function $s_b(x)$ are

$$\text{simple poles} \quad : \quad x = i(mb + n/b + Q/2), \quad (\text{C.1})$$

$$\text{zeros} \quad : \quad x = -i(mb + n/b + Q/2), \quad (\text{C.2})$$

for the non-negative integers $m, n = 0, 1, \dots$. The simple poles in the upper-half plane therefore come from the double-sine functions in the numerator of (3.5). Therefore by collecting the contribution from the pole $x_i^{mn} = -m_i - \mu_i/2 + i(mb + n/b)$, we obtain

$$\begin{aligned} Z[\sigma(\alpha), \rho(\alpha)] &= \sum_{i=1}^{N_f} \sum_{m,n=0}^{\infty} e^{-i\pi k(x_i^{mn})^2 + 2\pi i \xi x_i^{mn}} \\ &\times \prod_{\beta \neq \sigma(\alpha)} 2 \cosh \pi b(x_i^{mn} - \chi_\beta) \prod_{\beta \neq \rho(\alpha)} 2 \cosh \pi b^{-1}(x_i^{mn} - \chi'_\beta) \\ &\times \frac{\prod_{j \neq i}^{N_f} s_b(D_{ji}^{mn} + iQ/2)}{\prod_{j=1}^{N_f} s_b(C_{ji}^{mn} - iQ/2)} R^{mn}(b), \end{aligned} \quad (\text{C.3})$$

where

$$D_{ji}^{mn} = m_j - m_i + \mu_j/2 - \mu_i/2 + i(mb + n/b) = D_{ji} + i(mb + n/b), \quad (\text{C.4})$$

$$C_{ji}^{mn} = m_j - m_i - \mu_j/2 - \mu_i/2 + i(mb + n/b) = C_{ji} + i(mb + n/b), \quad (\text{C.5})$$

$$\begin{aligned} R^{mn}(b) &= 2\pi \text{Res}_{x=0} s_b(x + i(mb + n/b) + iQ/2) \\ &= \frac{(-1)^{mn} (-i)^{m+n} q^{m(m+1)/4} \tilde{q}^{n(n+1)/4}}{\prod_{k=1}^m (1 - q^k) \prod_{\ell=1}^n (1 - \tilde{q}^\ell)}. \end{aligned} \quad (\text{C.6})$$

Using the formulas (A.6) and (A.7), we can rewrite it into the following form

$$\begin{aligned}
& Z[\sigma(\alpha), \rho(\alpha)] \\
&= \sum_{i=1}^{N_f} \sum_{m,n=0}^{\infty} e^{-i\pi k(m_i+\mu_i/2)^2 - 2\pi i\xi(m_i+\mu_i/2)} (-1)^{(N-1)(m+n)} q^{-km^2/2} \tilde{q}^{-kn^2/2} \\
&\times e^{-2\pi bkm(m_i+\mu_i/2) - m\pi b \sum_j (\mu_j + iQ) - 2m\pi\xi b} e^{-2\pi b^{-1}kn(m_i+\mu_i/2) - n\pi/b \sum_j (\mu_j + iQ) - 2n\pi\xi/b} \\
&\times \frac{\prod_{j \neq i}^{N_f} s_b(D_{ji} + iQ/2)}{\prod_{j=1}^{N_f} s_b(C_{ji} - iQ/2)} \frac{\prod_j \prod_{k=1}^m (1 - q^{k-1} e^{-2\pi b C_{ji}}) \prod_{\ell=1}^n (1 - \tilde{q}^{\ell-1} e^{-2\pi b^{-1} C_{ji}})}{\prod_j \prod_{k=1}^m (1 - q^k e^{-2\pi b D_{ji}}) \prod_{\ell=1}^n (1 - \tilde{q}^{\ell} e^{-2\pi b^{-1} D_{ji}})} \\
&\times \prod_{\beta \neq \sigma(\alpha)} 2 \cosh \pi b(m_i + \mu_i/2 - imb + \chi_{\beta}) \prod_{\beta \neq \rho(\alpha)} 2 \cosh \pi b^{-1}(m_i + \mu_i/2 - in/b + \chi'_{\beta}),
\end{aligned} \tag{C.7}$$

for integral Chern-Simons coupling $k \in \mathbb{Z}$. Notice that the property $\cosh \pi b(x + in/b) = (-1)^n \cosh \pi b x$ prevents the partition function from mixing m and n sectors. This technical mechanism enables us to factorize the partition function into the vortex and anti-vortex blocks.

The abelianized partition function then takes the following factorized form

$$\begin{aligned}
& Z[\sigma(\alpha), \rho(\alpha)] \\
&= \sum_{i=1}^{N_f} z_{\text{cl}}^{(i)}(m, \mu) z_{\text{pert}}^{(i)}(m, \mu, b) z_V^{(i)}(m, \mu, q; \sigma(\alpha), \chi) \tilde{z}_V^{(i)}(m, \mu, \tilde{q}; \rho(\alpha), \chi').
\end{aligned} \tag{C.8}$$

The mechanism of this factorization is basically identical to that of abelian theory. Actually we find the following expressions for the classical and one-loop part are the same as those of $U(1)$ theory:

$$z_{\text{cl}}^{(i)}(m, \mu) = e^{-i\pi k(m_i+\mu_i/2)^2 - 2\pi i\xi(m_i+\mu_i/2)}, \tag{C.9}$$

$$z_{\text{pert}}^{(i)}(m, \mu, b) = \frac{\prod_{j \neq i}^{N_f} s_b(D_{ji} + iQ/2)}{\prod_{j=1}^{N_f} s_b(C_{ji} - iQ/2)}. \tag{C.10}$$

The vortex/anti-vortex part is given by

$$\begin{aligned}
& z_V^{(i)}(m, \mu, q; \sigma(\alpha), \chi) \\
&= \sum_{m=0}^{\infty} z_i^m (-1)^{(N-1)m} q^{-km^2/2} \prod_j \frac{\prod_{k=1}^m (1 - q^{k-1} e^{-2\pi b C_{ji}})}{\prod_{k=1}^m (1 - q^k e^{-2\pi b D_{ji}})} \\
&\times \prod_{\beta \neq \sigma(\alpha)} 2 \cosh \pi b(m_i + \mu_i/2 - imb + \chi_{\beta}),
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
& \tilde{z}_V^{(i)}(m, \mu, \tilde{q}; \rho(\alpha), \chi') \\
&= \sum_{n=0}^{\infty} \tilde{z}_i^n (-1)^{(N-1)n} \tilde{q}^{-kn^2/2} \prod_j \frac{\prod_{\ell=1}^n (1 - \tilde{q}^{\ell-1} e^{-2\pi b^{-1} C_{ji}})}{\prod_{\ell=1}^n (1 - \tilde{q}^{\ell} e^{-2\pi b^{-1} D_{ji}})} \\
&\quad \times \prod_{\beta \neq \rho(\alpha)} 2 \cosh \pi b^{-1} (m_i + \mu_i/2 - in/b + \chi'_{\beta}). \tag{C.12}
\end{aligned}$$

Here we introduced

$$z_i = e^{-2\pi b k(m_i + \mu_i/2) - \pi b \sum_j (\mu_j + iQ) - 2\pi \xi b} = e^{-2\pi b k(m_i + \mu_i/2) - 2\pi \xi_{\text{eff}} b}, \tag{C.13}$$

$$\tilde{z}_i = e^{-2\pi b^{-1} k(m_i + \mu_i/2) - \pi b^{-1} \sum_j (\mu_j + iQ) - 2\pi \xi b^{-1}} = e^{-2\pi b^{-1} k(m_i + \mu_i/2) - 2\pi \xi_{\text{eff}} b^{-1}}. \tag{C.14}$$

Since the full partition function consists of the abelianized partition functions (3.5), the full partition function takes the following factorized form

$$Z = \frac{1}{N!} \sum_{\{i_{\alpha}\}} Z_{\text{cl}}^{\{i_{\alpha}\}} Z_{\text{pert}}^{\{i_{\alpha}\}} Z_V^{\{i_{\alpha}\}} \tilde{Z}_V^{\{i_{\alpha}\}}, \tag{C.15}$$

where the summation is taken over the vortex sector which is labelled by the sequence of integers $\{i_1, \dots, i_N | i_{\alpha} = 1, \dots, N_f\}$. The classical and perturbative part of the partition functions are basically the products of over all the abelian contributions

$$Z_{\text{cl}}^{\{i_{\alpha}\}} = \prod_{\alpha=1}^N z_{\text{cl}}^{(i_{\alpha})}, \tag{C.16}$$

$$Z_{\text{pert}}^{\{i_{\alpha}\}} = \prod_{1 \leq \alpha < \beta \leq N} 4 \sinh(\pi b D_{i_{\alpha} i_{\beta}}) 4 \sinh(\pi b^{-1} D_{i_{\alpha} i_{\beta}}) \prod_{\alpha=1}^N z_{\text{pert}}^{(i_{\alpha})}. \tag{C.17}$$

The origin of the perturbative contribution from non-abelian vector multiplet

$$\prod_{1 \leq \alpha < \beta \leq N} 4 \sinh(\pi b D_{i_{\alpha} i_{\beta}}) 4 \sinh(\pi b^{-1} D_{i_{\alpha} i_{\beta}}), \tag{C.18}$$

will be clear in the following discussion. As we saw in section.3, we can factorize the perturbative part as follows

$$Z_{\text{cl}}^{\{i_{\alpha}\}}(\xi) Z_{\text{pert}}^{\{i_{\alpha}\}} = Z_{\text{cl}}^{\{i_{\alpha}\}}(\xi_{\text{eff}}) Z_{1\text{-loop}}^{\{i_{\alpha}\}} \tilde{Z}_{1\text{-loop}}^{\{i_{\alpha}\}}. \tag{C.19}$$

The vortexx/antivortex part is more complicated. Since the Cauchy formula involves a summation over the permutation, we have the following expression

$$Z_V^{\{i_{\alpha}\}} = \frac{\sum_{\sigma} (-1)^{\sigma} \prod_{\alpha=1}^N z_V^{(i_{\alpha})}(m, \mu, q; \sigma(\alpha), \chi)}{\prod_{1 \leq \alpha < \beta \leq N} 4 \sinh \pi b (-\chi_{\alpha} + \chi_{\beta}) \sinh(\pi b D_{i_{\alpha} i_{\beta}})}, \tag{C.20}$$

$$\tilde{Z}_V^{\{i_{\alpha}\}} = \frac{\sum_{\rho} (-1)^{\rho} \prod_{\alpha=1}^N \tilde{z}_V^{(i_{\alpha})}(m, \mu, \tilde{q}; \rho(\alpha), \chi')}{\prod_{1 \leq \alpha < \beta \leq N} 4 \sinh \pi b^{-1} (-\chi'_{\alpha} + \chi'_{\beta}) \sinh(\pi b^{-1} D_{i_{\alpha} i_{\beta}})}. \tag{C.21}$$

From the construction, the full partition function is independent of the choice of the auxiliary parameters χ and χ' . Actually, taking the summation over the permutations σ by using the Cauchy formula again, we get

$$\begin{aligned}
Z_V^{\{i_\alpha\}} &= \sum_{m_1, \dots, m_N=0}^{\infty} \frac{\prod_{\alpha=1}^N z_\alpha^{m_\alpha} (-1)^{(N-1)m_\alpha} q^{-km_\alpha^2/2}}{\prod_{1 \leq \alpha < \beta \leq N} 2 \sinh(\pi b D_{i_\alpha i_\beta})} \prod_{j, \alpha} \frac{\prod_{k=1}^{m_\alpha} (1 - q^{k-1} e^{-2\pi b C_{ji_\alpha}})}{\prod_{k=1}^{m_\alpha} (1 - q^k e^{-2\pi b D_{ji_\alpha}})} \\
&\times \frac{1}{\prod_{1 \leq \alpha < \beta \leq N} 2 \sinh \pi b (-\chi_\alpha + \chi_\beta)} \sum_{\sigma} (-1)^\sigma \prod_{\alpha=1}^N \prod_{\beta \neq \sigma(\alpha)} 2 \cosh \pi b (m_{i_\alpha} + \mu_{i_\alpha}/2 - i m_\alpha b + \chi_\beta) \\
&= \sum_{m_1, \dots, m_N=0}^{\infty} \prod_{\alpha=1}^N z_\alpha^{m_\alpha} (-1)^{(N-1)m_\alpha} q^{-km_\alpha^2/2} \prod_{\alpha=1}^N \prod_j \frac{\prod_{k=1}^{m_\alpha} (1 - q^{k-1} e^{-2\pi b C_{ji_\alpha}})}{\prod_{k=1}^{m_\alpha} (1 - q^k e^{-2\pi b D_{ji_\alpha}})} \\
&\times \prod_{\alpha < \beta} \frac{2 \sinh \pi b (D_{i_\alpha i_\beta} - i m_\alpha b + i m_\beta b)}{2 \sinh(\pi b D_{i_\alpha i_\beta})}. \quad (\text{C.22})
\end{aligned}$$

We can show that this vortex partition function is precisely the “K-theoretic” uplift of the vortex partition function for $U(N)$ gauge theory with N_f antifundamentals and $N_f - N$ fundamentals which was derived by Shadchin [10]. Let us check this agreement. With some algebra, the holomorphic block takes the form

$$\begin{aligned}
Z_V^{\{i_\alpha\}} &= \sum_{m_1, \dots, m_N=0}^{\infty} \prod_{\alpha=1}^N z_\alpha^{m_\alpha} (-1)^{(N-1)m_\alpha} q^{-km_\alpha^2/2 + N_f m_\alpha/2} e^{\pi b m_\alpha \sum_j \mu_j} \\
&\times \prod_{\alpha < \beta} \frac{2 \sinh \pi b (D_{i_\alpha i_\beta} - i m_\alpha b + i m_\beta b)}{2 \sinh(\pi b D_{i_\alpha i_\beta})} \prod_{\alpha=1}^N \prod_{i_\alpha=1}^{N_f} \frac{\prod_{l=1}^{m_\alpha} 2 \sinh \pi b (C_{j_\alpha i_\alpha} + i(l-1)b)}{\prod_{l=1}^{m_\alpha} 2 \sinh \pi b (D_{j_\alpha i_\alpha} + i l b)} \quad (\text{C.23})
\end{aligned}$$

As explained in Appendix.B of [21]³, we can rewrite the partition function into the following form

$$\begin{aligned}
Z_V^{\{i_\alpha\}} &= \sum_{m_1, \dots, m_N=0}^{\infty} \prod_{\alpha=1}^N ((-1)^N e^{\pi b \sum \mu_j} q^{N_f/2} z_\alpha)^{m_\alpha} q^{-km_\alpha^2/2} \\
&\times \prod_{\alpha=1}^N \prod_{l=1}^{m_\alpha} \frac{\prod_{j=1}^{N_f} 2 \sinh \pi b (C_{ji_\alpha} + i(l-1)b)}{\prod_{\beta=1}^N 2 \sinh \pi b (D_{i_\alpha i_\beta} + i(l-1-m_\alpha)b) \prod_{j=1, j \notin \{i_\alpha\}}^{N_f} 2 \sinh \pi b (D_{ji_\alpha} + i l b)}. \quad (\text{C.24})
\end{aligned}$$

Meanwhile, with the parametrization (3.12), (3.13) and (3.14), we can rewrite the arguments of the sinh factors

$$C_{ji_\alpha} = M_j - a_\alpha, \quad D_{ji_\alpha} = \bar{M}_j - a_\alpha + i b, \quad D_{i_\alpha i_\beta} = a_\alpha - a_\beta. \quad (\text{C.25})$$

³Such computation is ubiquitous in the study of the Nekrasov partition functions. See also [37] for instance.

Then, it is easy to see that this is precisely the vortex partition function of the above-mentioned $U(N)$ theory on $S^1 \times \mathbb{R}^2$ whose chiral multiplets have masses M_j and \tilde{M}_j .

Appendix C.2 chiral theory

Let us move on to the chiral gauge theory. Our theory is $U(N)$ theory with $2N_F$ fundamental chiral multiplets, and the axial masses satisfy $\sum_{i=1}^{2N_F} m_j = 0$.

After using the Cauchy formula and performing the integral using the residues at the simple poles $x_i^{mn} = -\mu_i/2 + i(mb + n/b)$, the abelianized partition function is

$$\begin{aligned} Z[\sigma(\alpha), \rho(\alpha)] &= \sum_{i=1}^{N_f} \sum_{m,n=0}^{\infty} e^{-i\pi k(x_i^{mn})^2 + 2\pi i \xi x_i^{mn}} \\ &\times \prod_{\beta \neq \sigma(\alpha)} 2 \cosh \pi b(x_i^{mn} - \chi_\beta) \prod_{\beta \neq \rho(\alpha)} 2 \cosh \pi b^{-1}(x_i^{mn} - \chi'_\beta) \\ &\times \prod_{j \neq i}^{2N_f} s_b(E_{ji}^{mn} + iQ/2) R^{mn}(b), \end{aligned} \quad (\text{C.26})$$

where $E_{ji} = \mu_j/2 - \mu_i/2$. With the formulas (A.6) and (A.7), we can rewrite it as

$$\begin{aligned} Z[\sigma(\alpha), \rho(\alpha)] &= \sum_{i=1}^{N_f} \sum_{m,n=0}^{\infty} e^{-i\pi k(\mu_i/2)^2 - 2\pi i \xi(\mu_i/2)} (-1)^{(N-1)(m+n)} q^{-km^2/2} \tilde{q}^{-kn^2/2} \\ &\times e^{-2\pi mb(\xi_{\text{eff}} + \frac{\mu_i}{2} k_{\text{eff}} - 2\frac{iN_f Q}{2})} e^{-2\pi nb^{-1}(\xi_{\text{eff}} + \frac{\mu_i}{2} k_{\text{eff}} - 2\frac{iN_f Q}{2})} \\ &\times \frac{\prod_{j \neq i}^{N_f} s_b(E_{ji} + iQ/2)}{\prod_j \prod_{k=1}^m (1 - q^k e^{-2\pi b E_{ji}}) \prod_{\ell=1}^n (1 - \tilde{q}^\ell e^{-2\pi b^{-1} E_{ji}})} \\ &\times \prod_{\beta \neq \sigma(\alpha)} 2 \cosh \pi b(\mu_i/2 - imb + \chi_\beta) \prod_{\beta \neq \rho(\alpha)} 2 \cosh \pi b^{-1}(\mu_i/2 - in/b + \chi'_\beta) \end{aligned} \quad (\text{C.27})$$

for integral Chern-Simons coupling $k \in \mathbb{Z}$. From this expression we can obtain the following factorization as in the case of the vector-like theory

$$Z = \frac{1}{N!} \sum_{\{i_\alpha\}} Z_{\text{cl}}^{\{i_\alpha\}} Z_{\text{pert}}^{\{i_\alpha\}} Z_V^{\{i_\alpha\}} \tilde{Z}_V^{\{i_\alpha\}}. \quad (\text{C.28})$$

The holomorphic block for this chiral theory is

$$\begin{aligned} Z_V^{\{i_\alpha\}} &= \sum_{m_1, \dots, m_N=0}^{\infty} \prod_{\alpha=1}^N z_\alpha^{m_\alpha} (-1)^{(N-1)m_\alpha} q^{-km_\alpha^2/2} \prod_{j=1}^{2N_f} \prod_{\alpha=1}^N \frac{1}{\prod_{l=1}^{m_\alpha} (1 - q^l e^{-2\pi b(\mu_j/2 - \mu_{i_\alpha}/2)})} \\ &\times \prod_{\alpha < \beta} \frac{2 \sinh \pi b(\mu_{i_\alpha}/2 - \mu_{i_\beta}/2 - im_\alpha b + im_\beta b)}{2 \sinh \pi b(\mu_{i_\alpha}/2 - \mu_{i_\beta}/2)}. \end{aligned} \quad (\text{C.29})$$

where

$$z_\alpha = e^{-2\pi b \left(k_{\text{eff}} \frac{\mu_{i_\alpha}}{2} + \xi_{\text{eff}} - \frac{iQ N_f}{2} \right)}, \quad \tilde{z}_\alpha = e^{-2\pi b^{-1} \left(k_{\text{eff}} \frac{\mu_{i_\alpha}}{2} + \xi_{\text{eff}} - \frac{iQ N_f}{2} \right)}. \quad (\text{C.30})$$

Using the result in Appendix B of [21], we can rewrite it into the following form

$$Z_V^{\{i_\alpha\}} = \sum_{m_1, \dots, m_N=0}^{\infty} \prod_{\alpha=1}^N \left((-1)^N q^{N_f} z_\alpha \right)^{m_\alpha} q^{-k m_\alpha^2 / 2} \\ \times \prod_{\alpha=1}^N \prod_{l=1}^{m_\alpha} \frac{1}{\prod_{\beta=1}^N 2 \sinh \pi b (E_{i_\alpha i_\beta} - i l b) \prod_{j=1, \neq \{i_\alpha\}}^{2N_f} 2 \sinh \pi b (E_{j i_\alpha} + i l b)}, \quad (\text{C.31})$$

and this is precisely the K-theoretic uplift of the vortex partition function for $U(N)$ theory with $2N_f - N$ fundamentals [10] .

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